GCE: 502, Linear Algebra
January 2018

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Exercise 1:
Let V be a finite-dimensional complex vector space and let $\tau : V \to V$ be a linear operator on V. Prove that τ has an eigenvector.

Exercise 2:
Let A be an $n \times n$ Hermitian matrix with largest eigenvalue λ_1. Let B be the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column of A. If μ_1 is the largest eigenvalue of B, prove that $\mu_1 \leq \lambda_1$.

Exercise 3:
Let V be the real inner product space of infinitely differentiable functions $f(t)$ on $[0, 1]$ satisfying $f(0) = f(1)$ endowed with the inner product

$$\langle f, g\rangle := \int_0^1 f(t)g(t)\, dt \ .$$

The differential operator $\tau = \frac{d}{dt}$ can be naturally viewed as a linear operator on V. Find an expression for the *adjoint operator* τ^* of τ.

Exercise 4:
In this exercise all matrices and vectors will be over $\mathbb{R}$. Let A be an n by p matrix and B be an invertible p by p matrix. Let b be a column vector in $\mathbb{R}^n$ and c be a scalar in $(0, \infty)$.
(i). Show that the equation

$$(A^tA + cB^tB)x = A^tb \tag{1}$$

is uniquely solvable and that its solution x_0 minimizes $||Ay-b||^2+c||By||^2$ for column vectors y in $\mathbb{R}^p$.
(ii). Let x_0 solve equation (1). Find (with proof) the limit of x_0 as c tends to infinity.
(iii). Let x_0 solve equation (1). Find (with proof) the limit of Ax_0 as c tends to zero.

Exercise 5:
Let M be an $n \times n$ matrix whose entries form an arithmetic progression, i.e.

$$M_{i,j} = a + (n(i-1) + (j-1))d,$$

for $1 \leq i, j \leq n$. Compute the determinant of M.

Exercise 6:
(i). Find a complex 4 by 4 matrix M such that $M^4 \neq I_4$ but $M^8 = I_4$.
(ii). Prove or disprove: there is a 4 by 4 integer matrix M such that $M^4 \neq I_4$ but $M^8 = I_4$.